

DIMENSIONS OF HIGHER EXTENSIONS FOR SL_2

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ABSTRACT. We analyse the recursive formula found for various Ext groups for $SL_2(k)$, k a field of characteristic p , and derive various generating functions for these groups. We use this to show that the growth rate for the cohomology of $SL_2(k)$ is at least exponential. In particular, $\max\{\dim \operatorname{Ext}_{SL_2(k)}^i(k, \Delta(a)) \mid a, i \in \mathbb{N}\}$ has (at least) exponential growth for all p . We also show that $\max\{\dim \operatorname{Ext}_{SL_2(k)}^i(k, \Delta(a)) \mid a \in \mathbb{N}\}$ for a fixed i is bounded.

INTRODUCTION

A very general open problem in the characteristic p representation theory of algebraic groups is to determine the higher extension groups $\operatorname{Ext}_G^q(M, N)$ where M, N are Weyl modules, or simple modules (or more generally if possible). In [7] the third author found recursive formula for many different Ext groups for modules in $SL_2(k)$, k an algebraically closed field of characteristic p . But no closed formula were found. More recently there has been interest in finding upper bounds for the dimensions of Ext groups (see for example [8]). Work of [10] applied the results of [7] to show that the growth rates of $\operatorname{Ext}_{SL_2(k)}^i(k, L)$, taken over all L a simple module and $i \in \mathbb{N}$ is at least exponential. Several years ago, just after [7] had been done, we had found some generating functions for the dimensions of Ext groups for Weyl modules, to investigate how large these could be, (mentioned in [2, section 6]). The recent work of Parshall and Scott in [8] and [9], and of Stewart [10], has encouraged us to polish this work up to explore further questions raised by these authors. In particular, using our generating functions, we have got an analogue of the exponential growth found by Stewart [10] for all primes, but using Weyl modules rather than simples. We also show that when we fix i that $\max\{\dim \operatorname{Ext}_{SL_2(k)}^i(k, \Delta(a)) \mid a \in \mathbb{N}\}$ is bounded (see section 7).

For prime 2 we have an explicit formula for the dimension of $\operatorname{Ext}_{SL_2(k)}^n(\Delta(0), \Delta(a))$. When a is odd this is zero by block considerations, so let $a = 2d$. We show that this is equal to the number of partitions (b_0, b_1, \dots, b_n) such that $\sum_i 2^{b_i} = d + 1$ (see corollary 3.2.2). These are compatible with Stewart's results.

For $p > 2$ the situation is more complicated. Our generating function $G(s)$ can be written as $\sum_{n \geq 0} z^n h_n(s)$ with $h_n(s)$ a power series, and the coefficient of s^d in $h_n(s)$ is the dimension of $\operatorname{Ext}_{SL_2(k)}^n(\Delta(0), \Delta(2d))$. We have a recursion for $sh_n(s)$, given in 4.5 and an algorithm which is described in 4.7. Exponential growth is established for $p = 2$ in Proposition 6.1.1 and for odd primes in Lemma 5.5.2 together with Lemma 5.6.2. That is, for a fixed prime, if we let d vary with n , then the dimensions of $\operatorname{Ext}_{SL_2(k)}^n(\Delta(0), \Delta(2d))$ grow exponentially.

1. PRELIMINARIES

1.1. Notation. We first briefly review some of the notation and definitions that we will use in this paper. The reader is referred to [4] for further information. We let $G = \mathrm{SL}_2(k)$ where k is an algebraically closed field of characteristic p , and $F : G \rightarrow G$ the corresponding Frobenius morphism. We may “twist” G -modules via this morphism. We let X^+ be the set of dominant weights which may be identified with \mathbb{N} , the non-negative integers.

For $\lambda \in X^+$, let k_λ be the one-dimensional module for B a suitable Borel, which has weight λ . We define $\nabla(\lambda) = \mathrm{Ind}_B^G(k_\lambda)$. This module has character given by Weyl’s character formula and has simple socle $L(\lambda)$, the irreducible G -module of highest weight λ . In the case of SL_2 all simples are known via Steinberg’s tensor product theorem. If we let E be the 2-dimensional natural module for $\mathrm{SL}_2(k)$, then $\nabla(\lambda) = S^\lambda E$, the λ th symmetric power of E . We will also use Weyl modules $\Delta(\lambda)$ which for our purposes can be either thought as divided powers, so $\Delta(\lambda) = D^\lambda E$ or as duals of induced modules: $\Delta(\lambda) = \nabla(\lambda)^*$, where $*$ is the usual k -linear dual.

The category of rational G -modules has enough injectives and so we may define $\mathrm{Ext}^*(-, -)$ as usual by using injective resolutions.

1.2. Background. Past work of the third author [5, 6], was concerned with finding explicit bounds on the global dimension of the Schur algebra associated to polynomial modules for $\mathrm{GL}_n(k)$. Of course it was known that such a bound should exist as the category of G -modules with bounded highest weight is an example of what is now known as a high weight category (or equivalently, is the module category of a quasi-hereditary algebra) and as such has finite global dimension. The work of [6] showed that for any algebraic group that

$$\mathrm{Ext}_G^m(L(w \cdot \lambda), L(v \cdot \lambda)) = \mathrm{Ext}_G^m(\nabla(w \cdot \lambda), \Delta(v \cdot \lambda)) = \begin{cases} 0 & \text{if } m > l(w) + l(v), \\ k & \text{if } m = l(w) + l(v) \end{cases}$$

where λ is in the interior of the fundamental alcove, $w \in W_p$, the affine Weyl group acting via the dot action on the dominant weights and $l : W_p \rightarrow \mathbb{N}$ is the usual length function on (the Coxeter group) W_p . We also have:

$$\mathrm{Ext}_G^m(\Delta(w \cdot \lambda), \Delta(v \cdot \lambda)) = \begin{cases} 0 & \text{if } m > l(v) - l(w), \\ k & \text{if } m = l(v) - l(w) \end{cases}$$

with the same notation as above.

For $\mathrm{SL}_2(k)$ this was proved more directly in [5] and can be summarised as follows:

$$\mathrm{Ext}_{\mathrm{SL}_2(k)}^m(L(pa+i), L(pb+j)) = \begin{cases} 0 & \text{if } m > a + b, \\ k & \text{if } m = a + b \text{ and } pa + i \text{ and } pb + j \text{ are in the same } W_p\text{-orbit} \end{cases}$$

where $a, b, i, j \in \mathbb{N}$ and $0 \leq i, j \leq p-2$. We also have:

$$\text{Ext}_{SL_2(k)}^m(\Delta(pa+i), \Delta(pb+j)) = \begin{cases} 0 & \text{if } m > b-a \\ k & \text{if } m = b-a \text{ and } pa+i \text{ and } pb+j \text{ are in the same } W_p\text{-orbit.} \end{cases}$$

NB: The condition for $pa+i$ and $pb+j$ to be in the same W_p orbit is that either $a-b$ is even and $i=j$ or $a-b$ is odd and $i=p-2-j$. The case that will be of most use in what follows is that when $pa+i=0$. Then $pb+j$ is in the same W_p -orbit ($=G$ -block) as 0 when b is even and $j=0$ or b is odd and $j=p-2$.

Thus we know that most of the Ext groups are zero and we also know what some of the lower dimensional groups between Weyl modules are for $SL_2(k)$ thanks to work of the first author [3], and of Cox and the first author [1]. We won't cite these results directly, but note that

$$\begin{aligned} \dim \text{Hom}_{SL_2(k)}(\Delta(pa+i), \Delta(pb+j)) &\leq 1, \\ \dim \text{Ext}_{SL_2(k)}^1(\Delta(pa+i), \Delta(pb+j)) &\leq 1 \end{aligned}$$

and

$$\dim \text{Ext}_{SL_2(k)}^2(\Delta(pa+i), \Delta(pb+j)) \leq 2.$$

This of course rather does beg the question, can we calculate the other Ext groups and can we find bounds on their dimensions?

The work of [7] allows us in theory to calculate these Ext groups but with recursive formula. While this is easy to program into a computer, this formulation has not proved useful so far for more theoretical results. This paper is our attempt to put the recursive formula of [7] into a form which will allow us (or others) to answer such questions as is $\dim \text{Ext}_{SL_2(k)}^m(\Delta(pa+i), \Delta(pb+j))$ bounded? If it isn't, what's the growth rate like? The rest of the paper is devoted to applying the theory of generating functions to the recursions in order to give partial answers to some of these questions.

2. SOME RECURSIONS

Henceforth all Ext groups will be over $SL_2(k)$ so we will drop the subscript. (The algebraic group in any case should be self-evident from the highest weights of the modules involved.)

We first note that for all p ,

$$\text{Ext}^m(\Delta(0), \Delta(0)) = 0 \text{ for } m \geq 1 \tag{2.0.1}$$

$$\text{Hom}(\Delta(0), \Delta(2b)) = \begin{cases} k & \text{if } b = p^r - 1, r \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.0.2}$$

This is well-known, see for example [3, (1.5)(3)], although it can also be derived using the recursions below.

2.1. The recursion for $p = 2$. When $p = 2$ we apply Corollary 4.2 (or Corollary 5.2) in [7], with $N = \Delta(0)$ which is fixed by the Frobenius twist: For $d \geq 1$ and $m \geq 1$ we have

$$\mathrm{Ext}^m(\Delta(0), \Delta(2b)) \cong \mathrm{Ext}^{m-1}(\Delta(0), \Delta(2(b-1))) \oplus \mathrm{Ext}^m(\Delta(0), \Delta(b-1)). \quad (2.1.1)$$

Note that when b is even, the weights 0 and $b-1$ are in different blocks and the second summand is zero.

2.2. The recursion for $p > 2$. By [7, Corollary 4.2 or 5.2] we have

$$\mathrm{Ext}^m(\Delta(0), \Delta(2bp)) \cong \mathrm{Ext}^{m-1}(\Delta(0), \Delta(2bp-2)) \quad (2.2.1)$$

and

$$\mathrm{Ext}^m(\Delta(0), \Delta(2bp-2)) \cong \mathrm{Ext}^{m-1}(\Delta(0), \Delta(2(b-1)p)) \oplus \mathrm{Ext}^m(\Delta(0), \Delta(2(b-1))).$$

3. THE GENERATING FUNCTION FOR $p = 2$

In this section k has characteristic 2.

3.1. We fix an integer $d \geq 0$. Set

$$\varepsilon(2d) := \sum_{m \geq 0} \dim \mathrm{Ext}^m(\Delta(0), \Delta(2d)) z^m$$

which is a polynomial in z .

We translate (2.1.1) into a recursion. Firstly, $\varepsilon(0) = 1$. Next we have:

$$\varepsilon(2d) = \begin{cases} z\varepsilon(2(d-1)) & \text{if } d \text{ even, } > 0; \\ z\varepsilon(2(d-1)) + \varepsilon(d-1) & \text{if } d \text{ odd.} \end{cases}$$

We define the generating function, $G(s)$, for the $\varepsilon(2d)$'s by

$$G(s) = \sum_{d \geq 0} s^d \varepsilon(2d).$$

We now give a functional equation satisfied by $G(s)$.

Lemma 3.1.1. *We have*

$$(1 - zs)G(s) = sG(s^2) + 1.$$

Proof. By substituting the recursion formula, we get

$$G(s) = 1 + \sum_{d \geq 1} z s^d \varepsilon(2(d-1)) + \sum_{d \text{ odd}} s^d \varepsilon(d-1) \quad (3.1.1)$$

We can write this as

$$\begin{aligned} G(s) &= 1 + sz \left(\sum_{d \geq 1} s^{d-1} \varepsilon(2(d-1)) \right) + sG(s^2) \\ &= 1 + szG(s) + sG(s^2). \end{aligned}$$

□

3.2. We would like to find the coefficient of z^m in $G(s)$. This is a power series in s , and the coefficient of s^d is $\dim \text{Ext}^m(\Delta(0), \Delta(2d))$. We write

$$G(s) = \sum_{m \geq 0} z^m g_m(s).$$

Assume first that $m = 0$. By equation (2.0.1) we know that

$$g_0(s) = 1 + s + s^3 + s^7 + \dots = \sum_{k \geq 0} s^{2^k - 1}$$

Note that this is consistent with setting $z = 0$ in equation (3.1.1).

Lemma 3.2.1. *We have the recursion*

$$g_n(s) - sg_{n-1}(s) = sg_n(s^2)$$

Furthermore for $n \geq 1$ we have

$$sg_n(s) = \sum_{k=0}^{\infty} s^{2^k} (s^{2^k} g_{n-1}(s^{2^k})).$$

Proof. Recall that $(1 - zs)G(s) = 1 + sG(s^2)$. The left hand side of this is

$$\sum_{n \geq 0} z^n g_n(s) - \sum_{n \geq 0} z^{n+1} sg_n(s)$$

and the right hand side is

$$1 + \sum_{n \geq 0} z^n sg_n(s^2).$$

Equating the coefficients of z^n gives the first part. For the second part we multiply with s and then substitute s^{2^k} . This gives:

$$\begin{aligned} sg_n(s) - s^2 g_n(s^2) &= s^2 g_{n-1}(s) \\ s^2 g_n(s^2) - s^4 g_n(s^4) &= s^4 g_{n-1}(s^2) \\ s^4 g_n(s^4) - s^8 g_n(s^8) &= s^8 g_{n-1}(s^4) \\ &\vdots \end{aligned}$$

Adding these together then gives the statement.

□

Corollary 3.2.2. *We can write*

$$sg_n(s) = \sum_{\beta} s^{2^{b_n} + 2^{b_{n-1}} + \dots + 2^{b_0}}$$

where the sum is over all partitions $\beta = (b_0, b_1, \dots, b_n)$ of m , m taken over all \mathbb{N} . In particular, $\dim \text{Ext}^n(k, \Delta(2d))$ is equal to the number of partitions (b_0, b_1, \dots, b_n) such that $\sum_i 2^{b_i} = d + 1$.

Proof. By induction on n we get from Lemma 3.2.1 that

$$sg_n(s) = \sum_{0 \leq k_i} s^{2^{k_0} + 2^{k_1 + k_0} + \dots + 2^{k_n + k_{n-1} + \dots + k_1 + k_0}}.$$

Substitute $b_n = k_0$, $b_{n-1} = k_0 + k_1$, and so on, until $b_0 = \sum_{i=0}^n k_i$. Then set $\beta = (b_0, b_1, \dots, b_n)$, this is then a partition; and all partitions with $n + 1$ parts occur.

The coefficient of s^d in $g_n(s)$ is equal to $\dim \text{Ext}^n(\Delta(0), \Delta(2d))$. This is the coefficient of s^{d+1} in $sg_n(s)$ and thus it is equal to the number of partitions β with $n + 1$ parts such that $2^{b_n} + 2^{b_{n-1}} + \dots + 2^{b_0} = d + 1$. \square

We also have a formula for $G(s)$ as a sum of rational functions. Although we will not apply it here, we include it for completeness.

Lemma 3.2.3. *Let $F(s) = \prod_{k=0}^{\infty} (1 - zs^{2^k})$. Then we have*

$$F(s)sG(s) - F(s^2)s^2G(s^2) = sF(s^2)$$

and

$$G(s) = (1 - zs)^{-1} \left(1 + \frac{s}{(1 - zs^2)} + \frac{s^3}{(1 - zs^2)(1 - zs^4)} + \dots \right)$$

Proof. By the definition we have $F(s) = (1 - zs)F(s^2)$. Multiplying the functional equation in Lemma 3.1.1 with s gives:

$$(1 - zs)sG(s) = s^2G(s^2) + s.$$

Multiplying this with $F(s^2)$ gives the first statement:

$$\begin{aligned} F(s)sG(s) &= (1 - zs)F(s^2)sG(s) \\ &= F(s^2)s^2G(s^2) + sF(s^2). \end{aligned}$$

We now have

$$\begin{aligned} F(s)sG(s) - F(s^2)s^2G(s^2) &= sF(s^2) \\ F(s^2)s^2G(s^2) - F(s^4)s^4G(s^4) &= s^2F(s^4) \\ F(s^4)s^4G(s^4) - F(s^8)s^8G(s^8) &= s^4F(s^8) \\ &\vdots \end{aligned}$$

We add these equations and get

$$F(s)sG(s) = \sum_{k=0}^{\infty} s^{2^k} F(s^{2^{k+1}}).$$

Hence we have

$$\begin{aligned} G(s) &= \frac{1}{s} \sum_0^{\infty} s^{2^k} \frac{F(s^{2^{k+1}})}{F(s)} \\ &= \frac{1}{s} \left[\frac{sF(s^2)}{F(s)} + \frac{s^2F(s^4)}{F(s)} + \dots \right] \\ &= \frac{1}{s} \left[\frac{s}{1-zs} + \frac{s^2}{(1-zs)(1-zs^2)} + \dots \right] \end{aligned}$$

which gives the second part of the statement. \square

4. THE GENERATING FUNCTION FOR $p > 2$

4.1. Let z be a variable, we define for a fixed even integer $2d$:

$$\varepsilon(2d) := \sum_{m \geq 0} \dim \text{Ext}^m(\Delta(0), \Delta(2d)) z^m$$

which is a polynomial. Then $\varepsilon(2d)$ can only be non-zero for $2d$ where $\Delta(2d)$ is in the principal block, that is $2d$ of the form $2kp$ or $2kp - 2$.

We translate (2.2.1) into a recursion for the ε . First, we have the initial conditions:

$$\varepsilon(0) = 1 \quad \text{and} \quad \varepsilon(2p - 2) = 1 + z.$$

Next, we have the following recursions

$$\varepsilon(2kp) = z\varepsilon(2kp - 2) \quad (k \geq 1) \tag{4.1.1}$$

$$\varepsilon(2kp - 2) = z\varepsilon(2(k - 1)p) + \varepsilon(2(k - 1)) \quad (k \geq 1). \tag{4.1.2}$$

4.2. Let s be a variable, and set

$$G_0(s) := \sum_{k=0}^{\infty} \varepsilon(2kp) s^{kp}, \quad G_1(s) := \sum_{k=1}^{\infty} \varepsilon(2kp - 2) s^{kp-1}$$

and $G(s) = G_0(s) + G_1(s)$, the generating function for the ε 's. From (4.1.1) and (4.1.2) we get the following recursions

$$G_0(s) = 1 + zsG_1(s)$$

$$G_1(s) = zs^{p-1}G_0(s) + s^{p-1}G(s^p).$$

We write this in vector form. Let $\Phi(s) := \begin{pmatrix} G_0(s) \\ G_1(s) \end{pmatrix}$, then

$$\Phi(s) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & zs \\ zs^{p-1} & 0 \end{pmatrix} \Phi(s) + s^{p-1} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \end{pmatrix} \Phi(s^p) \tag{4.2.1}$$

Lemma 4.2.1. *The generating function satisfies the functional equation*

$$(1 - z^2 s^p)G(s) = (1 + z s^{p-1}) + s^{p-1}(1 + zs)G(s^p).$$

Proof. Set $u := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $v := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and set $w = u + v$. Then define

$$A(s) := \begin{pmatrix} 0 & s \\ s^{p-1} & 0 \end{pmatrix}.$$

With this, (4.2.1) becomes

$$(1 - zA(s))\Phi(s) = u + s^{p-1}vw^T\Phi(s^p). \quad (4.2.2)$$

Now, $A(s)^2 = s^p I$, and hence

$$(1 + zA(s))(1 - zA(s)) = (1 - z^2 s^p)I.$$

Using (4.2.2)

$$(1 - z^2 s^p)\Phi(s) = (1 + zA(s))(u + s^{p-1}vw^T\Phi(s^p)).$$

Now note that $G(s) = w^T\Phi(s)$, so we get (after premultiplying with w^T)

$$(1 - z^2 s^p)G(s) = w^T(1 + zA(s))u + s^{p-1}w^T(1 + zA(s))vG(s^p) \quad (4.2.3)$$

Now $w^T(1 + zA(s))u = 1 + zs^{p-1}$ and $w^T(1 + zA(s))v = 1 + zs$. With this, (4.2.3) becomes the stated formula. \square

Remark 4.2.2. Consider the generating function we had for $p = 2$, in Lemma 3.1.1. If we multiply both sides in that formula by $(1 + zs)$ and set $p = 2$ then we get the same as in 4.2.1.

By iteration this gives a formula for $G(s)$.

4.3. We define functions $h_n(s)$ via

$$G(s) = \sum_{n \geq 0} h_n(s)z^n.$$

We are interested in the $h_n(s)$ as the coefficient of s^r in $h_n(s)$ is equal to the dimension of $\text{Ext}^n(\Delta(0), \Delta(2r))$. First we have from section 2 that

$$h_0(s) = 1 + s^{p-1} + s^{p^2-1} + \dots = s^{-1} \sum_{k=0}^{\infty} s^{p^k}.$$

4.4. We now calculate $h_1(s)$. By Lemma 4.2.1 we have that

$$h_1(s) = s^{p-1} + s^p h_0(s^p) + s^{p-1} h_1(s^p)$$

and we write this in the form

$$s h_1(s) - s^p h_1(s^p) = s^p + s^{p+1} h_0(s^p). \quad (4.4.1)$$

For each $k \geq 0$ we substitute s^{p^k} into s , which gives

$$s^{p^k} h_1(s^{p^k}) - (s^{p^k})^p h_1((s^{p^k})^p) = (s^{p^k})^p + (s^{p^k})^{p+1} h_0((s^{p^k})^p).$$

Adding all these equations, most of the terms on the left hand side cancel, and we get

$$\begin{aligned} s h_1(s) &= \sum_{k \geq 0} s^{p^{k+1}} + \sum_{k \geq 0} s^{p^k} (s^{p^{k+1}} h_0(s^{p^{k+1}})) \\ &= \sum_{k \geq 0} s^{p^{k+1}} + \sum_{k \geq 0} s^{p^k} \left(\sum_{r \geq 0} (s^{p^{k+1}})^{p^r} \right). \end{aligned}$$

So we have

$$s h_1(s) = \sum_{k_1 \geq 0} s^{p^{k_1+1}} + \sum_{k_i \geq 0} s^{p^{k_1} + p^{k_1+1} + k_0 + 1} \quad (4.4.2)$$

and this agrees with the global description given in [3, section 3.5].

4.5. We now note some important conventions.

- (1) We always label the exponents so that $k_i \geq 0$.
- (2) For $s h_n(s)$, there will be in each sum one new parameter, which we call k_n , and we keep the names of the exponents from previous sums as they were. This allows us to keep track of where the parameters came from.

Let $n \geq 2$. Then we get from Lemma 4.2.1 that

$$h_n(s) - s^p h_{n-2}(s) = s^{p-1} h_n(s^p) + s^p h_{n-1}(s^p)$$

that is

$$s h_n(s) - s^p h_n(s^p) = s^{p+1} h_{n-2}(s) + s^{p+1} h_{n-1}(s^p)$$

With the method as in 4.4 we get

$$s h_n(s) = \sum_{k_n \geq 0} s^{p^{k_n+1}} [s^{p^{k_n}} h_{n-2}(s^{p^{k_n}})] + \sum_{k_n \geq 0} s^{p^{k_n}} [s^{p^{k_n+1}} h_{n-1}(s^{p^{k_n+1}})] \quad (4.5.1)$$

4.6. Assume $n = 2$. We calculate the first sum in (4.5.1) with $n = 2$. This is

$$\sum_{k_2 \geq 0} s^{p^{k_2+1}} \left[\sum_{k_0 \geq 0} (s^{p^{k_2}})^{p^{k_0}} \right] = \sum_{k_i \geq 0} s^{p^{k_2+1} + p^{k_2+k_0}}.$$

For the second sum, consider

$$s^{p^{k_2+1}} h_1(s^{p^{k_2+1}}) = \sum_{k_1 \geq 0} (s^{p^{k_2+1}})^{p^{k_1+1}} + \sum_{k_1, k_0 \geq 0} (s^{p^{k_2+1}})^{p^{k_1} + p^{k_1+k_0+1}}.$$

And therefore we have

$$sh_2(s) = \sum_{k_i \geq 0} s^{p^{k_2+1} + p^{k_2+k_0}} + \sum_{k_i \geq 0} s^{p^{k_2} + p^{k_2+k_1+2}} + \sum_{k_i \geq 0} s^{p^{k_2} + p^{k_2+k_1+1} + p^{k_2+k_1+k_0+2}}.$$

We now demonstrate how we can use this sum to find possible dimensions for Ext groups. Clearly, any two terms in the second sum are distinct. As well, any two terms in the third sum are distinct. But consider the first sum.

- (1) Terms with $k_0 = 0$ and $k_0 = 2$ coincide. We can write the first sum as

$$s^{p+1} + 2 \sum_{k_2 \geq 1} s^{p^{k_2+1} + p^{k_2}} + \sum_{k_2 \geq 0, k_0 \neq 0, 2} s^{p^{k_2+1} + p^{k_2+k_0}}.$$

- (2) We compare terms from the first sum and terms from the second sum. We use \tilde{k}_i for exponents in the second sum. Suppose

$$p^{\tilde{k}_2} (1 + p^{\tilde{k}_1+2}) = p^{k_2} (p + p^{k_0})$$

Then we must have $k_0 \geq 1$ and $\tilde{k}_2 = k_2 + 1$. Furthermore $k_0 - 1 = \tilde{k}_1 + 2$. Conversely if $k_0 = \tilde{k}_1 + 3$ and $k_2 = \tilde{k}_2 - 1$ then the terms are equal. So the conditions for equality are precisely

$$k_0 \geq 3, \quad \tilde{k}_2 \geq 1$$

Note that since $k_0 \geq 3$, this does not interfere with the equalities in the first sum.

These results are consistent with the result in [1, section 5]. The parameters with $\dim \text{Ext}^2(\Delta(0), \Delta(t)) = 2$ are precisely $t = 2(p^{u+1} + p^{u+1+a} - 1)$ where $a \geq 1$ and $u \geq 0$ and gives the set denoted by $\Psi^{2,2}(0)$ in [1, section 5]. The ones with $a = 1$ come from coincidences in the first sum. The ones with $a \geq 2$ are coincidences from the first and the second sum.

4.7. **Towards general n .** First we simplify notation. Take the formula for $sh_n(s)$, it has several sums, labelled by tuples of k_i and positive integers. To keep track over the sums, it suffices to write down the exponents of p occurring. As an example, take formula (4.4.2) for $sh_1(s)$. We label the first sum by

$$(k_1 + 1)$$

and we label the second sum by

$$(k_1, k_1 + k_0 + 1).$$

We always have $k_i \geq 0$, and the coordinates of the tuples will be (linear) functions in the k_i which we denote by $m_i = m_i(k_1, k_1, \dots, k_j)$. So in the above example, we write the second sum as

$$(m_1, m_2) \equiv \sum_{k_i \geq 0} s^{p^{m_1} + p^{m_2}}$$

where $m_1 = k_1$ and $m_2 = k_1 + k_0 + 1$.

In general we use therefore as a shorthand notation

$$(m_1, m_2, \dots, m_t) \equiv \sum_{k_i \geq 0} s^{p^{m_1} + p^{m_2} + \dots + p^{m_t}}$$

where the m_j are (linear) functions of the k_i . Then each $sh_n(s)$ is identified with a list of such (m_1, \dots, m_t) .

Example 4.7.1. With this notation,

- (1) $sh_0(s)$ is identified with (k_0) .
- (2) $sh_1(s)$ is identified with the list

$$(k_1 + 1), (k_1, k_1 + k_0 + 1).$$

- (3) $sh_2(s)$ is identified with

$$(k_2 + 1, k_2 + k_0), (k_2, k_2 + k_1 + 2), (k_2, k_2 + k_1 + 1, k_2 + k_1 + k_0 + 2).$$

Let $n \geq 2$. Then $sh_n(s)$ is obtained from $sh_{n-2}(s)$ and $sh_{n-1}(s)$ as described using (4.5.1). The shorthand notation as above allows us to write down $sh_n(s)$.

- (a) Suppose (m_1, \dots, m_t) labels a sum in $sh_{n-2}(s)$. This then leads to one sum in $sh_n(s)$, and it has label

$$(k_n + 1, k_n + m_1, \dots, k_n + m_t).$$

- (b) Now suppose (f_1, \dots, f_u) labels a sum in $sh_{n-1}(s)$. This then leads to one sum in $sh_n(s)$, and it has label

$$(k_n, k_n + f_1 + 1, \dots, k_n + f_u + 1).$$

For example, the list in Example 4.7.1 (3) is obtained from the lists in Example 4.7.1 (1) and (2) by this process.

This gives a complete algorithm for writing down $sh_s(n)$ in general.

5. EXPONENTIAL GROWTH FOR $p > 2$

For this section, p will be a fixed odd prime.

5.1. We first present an example to show how the labels get more complicated for degree $n = 3$.

Example 5.1.1. Consider $sh_3(s)$. The labels for $sh_3(s)$ which come from $sh_1(s)$ are

- (a) $(k_3 + 1, k_3 + k_1 + 1) = k_3(1, 1) + k_1(0, 1) + (1, 1)$,
- (b) $(k_3 + 1, k_3 + k_1, k_3 + k_1 + k_0 + 1) = k_3(1, 1, 1) + k_1(0, 1, 1) + k_0(0, 0, 1) + (1, 0, 1)$.

The labels for $sh_3(s)$ which come from $sh_2(s)$ are

- (c) $(k_3, k_3 + k_2 + 2, k_3 + k_2 + k_0 + 1) = k_3(1, 1, 1) + k_2(0, 1, 1) + k_0(0, 0, 1) + (0, 2, 1)$,
- (d) $(k_3, k_3 + k_2 + 1, k_3 + k_2 + k_1 + 3) = k_3(1, 1, 1) + k_2(0, 1, 1) + k_1(0, 0, 1) + (0, 1, 3)$,
- (e) $(k_3, k_3 + k_2 + 1, k_3 + k_2 + k_1 + 2, k_3 + k_2 + k_1 + k_0 + 3)$
 $= k_3(1, 1, 1, 1) + k_2(0, 1, 1, 1) + k_1(0, 0, 1, 1) + k_0(0, 0, 0, 1) + (0, 1, 2, 3)$.

5.2. We can write down $sh_4(s)$ by first taking $sh_2(s)$, and replace a label of length r by the label of length $r + 1$, obtained by adding

$$k_4(1, 1, \dots, 1) + (1, 0, 0, \dots, 0);$$

and then taking $sh_3(s)$, and replace a label of length r by the label of length $r + 1$ obtained by adding

$$k_4(1, 1, \dots, 1) + (0, 1, \dots, 1).$$

Similarly we can write down a list of labels for $sh_n(s)$ from those of $sh_{n-2}(s)$ and of $sh_{n-1}(s)$. Note that each label of length $r + 1$ say for $sh_n(s)$ is of the form

$$k_n(1, 1, \dots, 1) + k_{i_1}(0, 1, 1, \dots, 1) + \dots + k_{i_r}(0, 0, \dots, 0, 1) + (a_0, \dots, a_r),$$

where $n > i_1 > i_2 > \dots > i_r \geq 0$ and where the a_i are non-negative integers $\leq n$.

5.3. Let t_n be the number of labels for $sh_n(s)$, then we have the recursion

$$t_n = t_{n-1} + t_{n-2}$$

and $t_n = F_{n+1}$, the $(n + 1)$ -th Fibonacci number. This shows that the number of labels grows exponentially, as the Fibonacci sequence is known to have exponential growth.

5.4. Now we find the length of the labels and the number of labels for some specific length. This will show that for each n there is a number v_n such that the number of labels of length v_n grows polynomially.

For each n , let $\mathcal{L}(n)$ be the set of lengths for the labels. So we have

$$\mathcal{L}(0) = \{1\}, \quad \mathcal{L}(1) = \{1, 2\}, \quad \mathcal{L}(2) = \{2, 3\}$$

Lemma 5.4.1. (a) $\mathcal{L}(2t - 1) = \{t, t + 1, \dots, 2t\}$

(b) $\mathcal{L}(2t) = \{t + 1, t + 2, \dots, 2t + 1\}$

Proof. Induction on n . We always have

$$\mathcal{L}(n) = \{x + 1 : x \in \mathcal{L}(n-1) \cup \mathcal{L}(n-2)\} \quad (5.4.1)$$

Let $t = 1$, we have $\mathcal{L}(1) = \{1, 2\}$. As well $\mathcal{L}(2) = \{2, 3\}$, as stated.

Assume true for t , then by the general reduction

$$\mathcal{L}(2(t+1)-1) = \{t+1, t+2, \dots, 2t+1\} \cup \{t+2, t+3, \dots, 2t+2\}$$

with smallest $t+1$ and largest $2(t+1)$ as stated. Next we have

$$\mathcal{L}(2(t+1)) = \{t+2, t+3, \dots, 2t+2\} \cup \{t+2, t+3, \dots, 2t+3\}$$

with smallest $(t+1)+1$ and largest $2(t+1)+1$, as required. \square

The largest length of labels in $\mathcal{L}(n)$ is therefore $n+1$.

5.5. We fix n , and count labels of a given length.

Lemma 5.5.1. (a) *There is one label of length $n+1$.*

(b) *The number of labels of length $n-i$ is $\binom{n-i}{i+1}$ for $i = 0, 1, \dots$*

Proof. (a) We see this from (5.4.1) above, by induction.

(b) We use induction on n . The cases $n = 1, 2, 3$ are clear from the examples. For the inductive step, by (5.4.1), the number of labels of length $n-i$ in degree n is equal to

$$\# \text{ labels of length } n-(i+1) \text{ in degree } n-2 + \# \text{ labels of length } n-(i+1) \text{ in degree } n-1.$$

By the inductive hypothesis, this is

$$\binom{n-(i+1)}{i} + \binom{n-(i+1)}{(i+1)} = \binom{n-i}{i+1}. \quad \square$$

Lemma 5.5.2. *Fix n and p an odd prime. Let $v = v_k$ be the number of labels of length k . Then there is a weight $m = p^{b_1} + p^{b_2} + \dots + p^{b_k}$ such that*

$$\dim \text{Ext}^n(\Delta(0), \Delta(2(m-1))) \geq v$$

Proof. Each label is of the form

$$\sum_j k_{i_j} (0, \dots, 0, 1, 1, \dots, 1) + (a_1, a_2, \dots, a_k)$$

(as described in 5.2 above). We can find $\beta := (b_1, b_2, \dots, b_k)$ such that $0 \leq b_1 - a_1 \leq b_2 - a_2 \leq \dots \leq b_k - a_k$ for each (a_1, \dots, a_k) occurring in the list of labels. For such β , we can in each label find a solution for the $k's$.

Let $m = \sum p^{b_i}$. Then the coefficient of s^{m-1} in $sh_n(s)$ is at least v . Hence the claim follows, with this m . \square

Since the number of labels of the same length k grows polynomially this shows that we have at least polynomial growth in n of $\dim \text{Ext}^n(k, \Delta(a))$. The degree of these polynomials is unbounded however.

We present a small example for a 3-dimensional Ext^3 space:

Example 5.5.3. Consider prime $p = 3$ and weight 76. We have $76 = 2 \cdot 38$. Using the function $sh_3(s)$, the dimension of $\text{Ext}^3(\Delta(0), \Delta(76))$ is equal to the number of times s^{38+1} occurs. Now,

$$39 = 27 + 9 + 3 = 3^3 + 3^2 + 3^1$$

We want to find all solutions for the k_i to get $\{3, 2, 1\}$.

The label in (a) has only two terms, so this does not contribute.

Consider the label in (b). The last entry is at least as big as the other two, so if there is a solution we must have

$$k_3 + k_1 + k_0 + 1 = 3$$

Then there are two possibilities,

- (i) $k_3 + 1 = 2$ and $k_3 + k_1 = 1$, or
- (ii) $k_3 + 1 = 1$ and $k_3 + k_1 = 2$.

In the first case, $(k_3, k_2, k_0) = (1, 0, 1)$. In the second case $(k_3, k_1, k_0) = (0, 2, 0)$.

Consider the label in (c). The first entry is at most as big as the others, so we must have $k_3 = 1$, and then there is a unique solution, $(k_3, k_2, k_0) = (1, 0, 0)$.

For label (d) there is no solution, and label (e) has too many terms so there is also no solution.

Thus, we have 3 possible solutions to get $\{3, 2, 1\}$ and thus $\dim \text{Ext}^3(k, \Delta(76)) = 3$.

5.6. We now show that the maximum of the number of the labels i.e. $\max_{a \in \mathbb{N}} \left\{ \binom{n-a}{a+1} \right\}$ grows exponentially.

Firstly we have:

Lemma 5.6.1.

$$\binom{n-a}{a+1} \geq \binom{n-a+1}{a} \text{ if } a < \left(\frac{1}{2} - \frac{\sqrt{5}}{10} \right) n - 1$$

and

$$\binom{n-a}{a+1} \leq \binom{n-a+1}{a} \text{ if } \frac{n}{2} \geq a > \left(\frac{1}{2} - \frac{\sqrt{5}}{10} \right) n.$$

Proof. For $a \leq \frac{n+1}{2}$

$$\binom{n-a}{a+1} \div \binom{n-a+1}{a} = \frac{(n-2a+1)(n-2a)}{(a+1)(n-a+1)}.$$

Now

$$(n-2a+1)(n-2a) - (a+1)(n-a+1) = n^5 - 5an + 5a^2 - 2a - 1 = 5a^2 - a(5n+2) + n^2 - 1.$$

Since n is fixed, we consider this as a parabolic in a . This has two roots and will be positive outside the roots and negative inside. The roots are not so easy to work out so we show that if $a = n \left(\frac{1}{2} - \frac{\sqrt{5}}{10} \right)$ then

$$5a^2 - a(5n+2) + n^2 - 1 = -n \left(1 - \frac{\sqrt{5}}{5} \right) - 1 < 0$$

for $n > 0$ as $1 > \frac{\sqrt{5}}{5}$. If $a = n \left(\frac{1}{2} - \frac{\sqrt{5}}{10} \right) n - 1$ then

$$5a^2 - a(5n+2) + n^2 - 1 = n \left(3 - \frac{2\sqrt{5}}{5} \right) + 6 > 0$$

for $n > 0$ as $3 > \frac{2\sqrt{5}}{5}$. So there is a root between these two values.

If $a = \frac{n}{2}$ then

$$5a^2 - a(5n+2) + n^2 - 1 = -\frac{n^2}{4} - n - 1 < 0$$

for $n > 0$ so the next root is past this value.

Thus

$$\binom{n-a}{a+1} \div \binom{n-a+1}{a} \begin{cases} > 1 & \text{if } a < \left(\frac{1}{2} - \frac{\sqrt{5}}{10} \right) n - 1 \\ < 1 & \text{if } \frac{n}{2} \geq a > \left(\frac{1}{2} - \frac{\sqrt{5}}{10} \right) n \end{cases} \quad \square$$

Thus the maximum value of $\binom{n-1}{a+1}$ for fixed n and varying a is achieved around $a = \left\lfloor \left(\frac{1}{2} - \frac{\sqrt{5}}{10} \right) n \right\rfloor$.

We now use Stirling's formula to estimate the growth rate of this in n . Note: since we are estimating growth rates and binomial coefficients can be defined for non-integers we are going to ignore the fact that a is not an integer in the next Lemma.

Lemma 5.6.2. *Let $a = \left(\frac{1}{2} - \frac{\sqrt{5}}{10} \right) n - 1$ then the growth rate of $\binom{n-a}{a+1}$ is exponential in n .*

Proof. Stirling's formula says $n! \sim \sqrt{2\pi n} \left(\frac{n}{e} \right)^n$. (The \sim denotes asymptotically equal.) Set $A := \frac{1}{2} - \frac{\sqrt{5}}{10}$ and $\bar{A} := \frac{1}{2} + \frac{\sqrt{5}}{10}$ so $A + \bar{A} = 1$. We have

$$\begin{aligned} \binom{n-a}{a+1} &= \frac{(n-An+1)!}{(An)!(n-2An+1)!} \\ &\sim \frac{1}{\sqrt{2\pi}} \sqrt{\frac{(\bar{A}n+1)}{(An)(\frac{\sqrt{5}}{5}n+1)}} \frac{(\bar{A}n+1)^{(\bar{A}n+1)}}{(An)^{(An)}(\frac{\sqrt{5}}{5}n+1)^{(\frac{\sqrt{5}}{5}n+1)}} \end{aligned}$$

set $C := \frac{\sqrt{5}}{5} = \bar{A} - A$:

$$= \frac{1}{\sqrt{2\pi}} \frac{(\bar{A}n+1)^{(\bar{A}n+\frac{3}{2})}}{(An)^{(An+\frac{1}{2})}(Cn+1)^{(Cn+\frac{3}{2})}} =: f(n)$$

So what growth rate does this function f have? For a crude approximation to first order for very large n , f is after setting $D := \frac{\bar{A}^{\frac{3}{2}}}{C^{\frac{3}{2}}A^{\frac{1}{2}}}$ and $E := \frac{\bar{A}^{\bar{A}}}{A^A C^C}$:

$$f(n) \approx \frac{1}{\sqrt{2\pi}} \frac{(\bar{A}n+\frac{3}{2})^{(\bar{A}n+\frac{1}{2})}}{(An)^{(An)}(Cn)^{(Cn+\frac{3}{2})}} = \frac{D}{\sqrt{2\pi}} E^n n^{((\bar{A}-A-C)n-\frac{1}{2})} = \frac{D}{\sqrt{2\pi}} E^n n^{-\frac{1}{2}}.$$

Which, as $E > 1$, means this has exponential growth in n . \square

Corollary 5.6.3. *We may find weights a_n such that the sequence $\dim \text{Ext}^n(k, \Delta(a_n))$ grows exponentially.*

6. EXPONENTIAL GROWTH FOR $p = 2$

We analyse the functions $sg_n(s)$.

6.1. In Corollary 3.2.2 we found an explicit formula for $\dim \text{Ext}^i(\Delta(0), \Delta(2d))$. We now compare this result to that found in Stewart.

Let $n = 2m$ where $m > 2$. In [10, Theorem 2] it is stated that

$$\dim \text{Ext}^{2m}(\Delta(0), L(2^{2m})) \geq 2^{m-1}.$$

In fact he proves that $\dim \text{Ext}^m(\Delta(0), L(2^m))$ is equal to the number of partitions of 1 into m powers of $\frac{1}{2}$, [10, section 2]. Let's call this number Π_{m-1} . Now we have that $\dim \text{Ext}^m(\Delta(0), \Delta(2^m))$ is the number of partitions $\beta = (b_0, b_1, b_2, \dots, b_m)$ with $m+1$ parts such that $2^{b_m} + 2^{b_{m-1}} + \dots + 2^{b_0} = 2^m + 1$. Note that both sides of this equation are 1 modulo 2 for $m \geq 1$. This means in particular that at least one of the b_i on the LHS is 1. Thus for this particular weight, the dimension is equal to the number of partitions $\beta' = (b_0, b_1, \dots, b_{m-1})$ with $2^{b_{m-1}} + 2^{b_{m-2}} + \dots + 2^{b_0} = 2^m$. Also note that $2^{b_i} \leq 2^m$, i.e. that $b_i \leq m$. Clearly,

$$2^{b_{m-1}} + 2^{b_{m-2}} + \dots + 2^{b_0} = 2^m \Leftrightarrow 2^{-(m-b_{m-1})} + 2^{-(m-b_{m-2})} + \dots + 2^{-(m-b_0)} = 1$$

I.e. the number of such partitions is exactly Π_{m-1} . That these numbers should be equal is not so suprising at least in one case. Namely from the structure of $\Delta(2d)$ for $d = 2^m$, $m \geq 1$ we can deduce that $\text{Ext}^{2m}(\Delta(0), L(2d)) \cong \text{Ext}^{2m}(\Delta(0), \Delta(2d))$. That is we use that the radical of $\Delta(2d)$ is isomorphic to a dual Weyl module $\nabla(2(d-1))$, and then note that $\text{Ext}^i(\Delta(0), \nabla(2(d-1))) = 0$. To see that the structure of $\Delta(2d)$ is as claimed note that we have a (well-known) short exact sequence

$$0 \rightarrow L(2^m - 1)^F \rightarrow \Delta(2^{m+1}) \rightarrow \Delta(2^m)^F \rightarrow 0$$

where F is the twist by the Frobenius morphism. The result then follows by induction and using that $\nabla(2^m - 2)$ is the only module with simple socle $L(2^m - 1)^F$ and the required character.

We can show that the dimensions of Ext spaces with similar parameters are also large. We have the following:

Proposition 6.1.1. *Let $t \geq 2$. Then the dimension of $\text{Ext}^{2t-2}(\Delta(0), \Delta(2 \cdot 2^t))$ is at least 2^{t-2} .*

This is similar to [10, Theorem 2] but with a different weight for the second Weyl module. We now give a proof in our setup.

Recall that the dimension of this Ext space is the number of expansions of $2^t + 1$ of length $2t - 2 + 1$, i.e. (assuming $t \geq 2$) equal to the number of expansions of 2^t of length $2t - 2$.

We define the length of the expansion $2^t = \sum_i 2^{b_i}$ to be the number of b_i occurring (including zeros).

Lemma 6.1.2. *Fix an integer $m \geq 4$. Let \mathcal{M}_r be the set of all expansions of 2^m of length r . Then*

$$|\mathcal{M}_{2m-2}| \leq |\mathcal{M}_{2m-1}|$$

Proof. We will define a map $\Psi : \mathcal{M}_{2m-2} \rightarrow \mathcal{M}_{2m-1}$ and show that it is 1-1.

We claim that b_1 (the first term of the partition) must be at least 2. If not then all b_i are at most 1 and it follows that

$$2^m = \sum 2^{b_i} \leq (2m-2) \cdot 2$$

and therefore $m \leq 3$, which contradicts the assumption. Hence define $\Psi(\sum_i 2^{b_i})$ to be the expansion obtained from $\sum 2^{b_i}$ when replacing 2^{b_1} by $2^{b_1-1} + 2^{b_1-1}$. We claim that the map is 1-1.

To do so, we write the partition $\beta = (b_i)$ as $(c_1^{r_1}, c_2^{r_2}, \dots, c_w^{r_w})$ with $c_1 > c_2 > \dots > c_w \geq 0$ (so that $r_i \geq 1$ and $\sum r_i = 2m-2$).

With this,

$$\Psi(c_1^{r_1}, c_2^{r_2}, \dots, c_w^{r_w}) = \begin{cases} (c_1^{r_1-1}, (c_1-1)^2, c_2^{r_2}, \dots) & c_1 - 1 > c_2 \\ (c_1^{r_1-1}, c_2^{r_2+2}, \dots) & c_1 - 1 = c_2 \end{cases}$$

Then it is easy to see that the map Ψ is 1-1 (note that $r_2 + 2 > 2$). □

Proof of Proposition 6.1.1. We show by induction on t that

$$\left. \begin{array}{l} \text{(a) The number of expansions of } 2^t \text{ of length } 2t-2 \text{ is } \geq 2^{t-2}, \text{ and} \\ \text{(b) The number of expansions of } 2^t \text{ of length } 2t-1 \text{ is } \geq 2^{t-2}. \end{array} \right\} (\mathbf{P}_t)$$

We start with $t = 2$. Then we have the expansions

$$4 = 2^1 + 2^1, \quad 4 = 2^1 + 2^0 + 2^0$$

and $2^{t-2} = 1$.

For the inductive step, assume (\mathbf{P}_m) holds for $2 \leq m < t$. Now consider expansions of 2^t of length $2t-2$, ie we want to verify part (a) of (\mathbf{P}_t) (then (b) will follow, by the Lemma).

- (1) We have $2^t = 2^{t-1} + 2^{t-1}$. For each expansion of 2^{t-1} of length $2t-3$ we have one of 2^t of length $2t-2$, by taking $b_1 = t-1$. Different such expansions of 2^{t-1} give different expansions of 2^t . Note that $2t-3 = 2(t-1)-1$. Hence by induction (using (b) for (P_{t-1})) the number of such expansions is $\geq 2^{t-3}$.
- (2) Next, we have $2^t = 2^{t-2} + 2^{t-2} + 2^{t-2} + 2^{t-2}$. For each expansion of 2^{t-2} of length $2t-5$ we get one expansion of 2^t , by taking $b_1 = b_2 = b_3 = 2^{t-2}$, and again different expansions of 2^{t-2} give different expansions of 2^t . Note that $2(t-2)-1 = 2t-5$ and hence by induction the number of such expansions is $\geq 2^{t-4}$.

We continue this way. In step s , we write

$$2^t = [2^{t-2} + 2^{t-2}] + [2^{t-3} + 2^{t-3}] + [2^{t-4} + 2^{t-4}] + \dots + [2^{t-s} + 2^{t-s} + 2^{t-s} + 2^{t-s}]$$

This is an expansion of length $2s$. We replace the last term 2^{t-s} by an expansion into $2(t-s) - 1$ terms and this gives an expansion of 2^t of length $2s - 1 + 2(t-s) - 1 = 2t - 2$. By induction the number of distinct such expansions is $\geq 2^{t-s-2}$.

These expansions are different from earlier expansions, as one sees by comparing the largest exponents.

At step $s = t - 2$ we replace 2^2 by expansions of length 3 and there is just 1 (and $1 = 2^0$). We do one more step, and replace 2 by expansions of length 1, and there is just one such expansion.

In total we have produced a list of distinct expansions of 2^t of length $2t - 2$, and the number of these expansions in our list is at least

$$2^{t-3} + 2^{t-4} + \dots + 2^2 + 2 + 1 + 1 = 2^{t-2}.$$

This proves the Proposition. \square

Remark 6.1.3. Using section 4 in [3] one gets dimensions of Ext groups between some Weyl modules for G , an arbitrary algebraic group. Namely, if λ, μ are dominant weights such that $\mu - \lambda$ differ by $2d\alpha$ where α is a simple root, and $d \geq 0$ then

$$\text{Ext}_G^m(\Delta(\lambda), \Delta(\mu)) \cong \text{Ext}_{SL_2}^m(\Delta(\langle \lambda, \alpha^\vee \rangle), \Delta(\langle \mu, \alpha^\vee \rangle))$$

This then shows that

$$\max\{\dim \text{Ext}_G^i(\Delta(\lambda), \Delta(\mu)) \mid \lambda, \mu \in X^+, i \in \mathbb{N}\}$$

has (at least) exponential growth for all p .

7. BOUNDING Ext

In this section we show that when we fix n and p , the dimensions of $\text{Ext}^n(k, \Delta(2d))$ have a bound independent of d .

7.1. We fix a prime p and consider the dimensions of $\text{Ext}^n(k, \Delta(2d))$ for a fixed n as d varies. We have two cases.

$p = 2$: The dimension is equal to the number of partitions (b_0, b_1, \dots, b_n) with $b_0 \geq b_1 \geq \dots \geq b_n \geq 0$ such that $\sum 2^{b_i} = d + 1$.

$p > 2$: The dimension is bounded above by the number of compositions (m_1, \dots, m_l) whose length l satisfies

- (i) if $n = 2t - 1$, then $t \leq l \leq 2t (= n + 1)$,
- (ii) if $n = 2t$, then $t + 1 \leq l \leq 2t + 1 (= n + 1)$

and such that $\sum p^{m_i} = d + 1$.

That is we need to understand the following set: Let $p \geq 2$ be a prime. Fix n , and take some number $l \leq n + 1$, then for each $d \in \mathbb{N}$, let

$$\mathcal{N}_d := \{(m_1, \dots, m_l) : 0 \leq m_j, \sum_j p^{m_j} = d + 1\}$$

We now show that this set is bounded.

Proposition 7.1.1. *The size $|\mathcal{N}_d|$ is bounded in terms of n , independent of d .*

Proof. Firstly we reformulate the problem. We can write

$$\sum_{j=1}^l p^{m_j} = \sum_{i=1}^r a_i p^{s_i}$$

where $0 < a_i$ is the number of times $m_j = s_i$, and $0 \leq s_1 < s_2 < \dots < s_r$. Then $\sum a_i = l$.

Fix (a_i) with $0 < a_i$ and $\sum a_i = l$, and define

$$\mathcal{N}_d(a_i) := \{(s_1, s_2, \dots, s_r) : \sum_{i=1}^r a_i p^{s_i} = d + 1, 0 \leq s_1 < s_2 < \dots < s_r\}$$

Then we have $\mathcal{N}_d = \bigcup \mathcal{N}_d(a_i)$, the union over all such (a_i) . For a fixed n , the number of ways writing $\sum a_i = l$ where $l \leq n + 1$ with $0 < a_i$ is bounded in terms of n . So it suffices to show the following.

We now claim that for any (a_i) the size of $\mathcal{N}_d(a_i)$ is bounded in terms of n , independent of d .

We show this by induction on n ; the case $n = 1$ is clear. For the inductive step, we consider first the case when p does not divide a_i for any i .

We first show that if p does not divide a_i for all i then $|\mathcal{N}_d(a_i)| \leq 1$.

Suppose the set is not empty. Assume (t_i) and (j_i) are in $\mathcal{N}_d(a_i)$, so that

$$d + 1 = \sum_i a_i p^{t_i} = \sum_i a_i p^{j_i}$$

and, say, $t_1 \leq j_1$. The p -part on the LHS is p^{t_1} and the p -part on the RHS is p^{j_1} . So $t_1 = j_1$. Subtract the lowest term, and repeat the argument. This shows $t_i = j_i$ for all i .

Now let's assume that at least one a_i is divisible by p . Write $a_i = b_i p^{u_i}$ with b_i not divisible by p , and then $u_i > 0$ for some i and therefore $\sum b_i < \sum a_i$, and as well $b_i > 0$. So we can then use the inductive hypothesis. To do so, we must arrange the sum by increasing p -powers:

If (j_1, \dots, j_r) is in $\mathcal{N}_d(a_i)$ then there is a permutation π of r such that

$$(\dagger) \quad u_{\pi(s)} + j_{\pi(s)} \leq u_{\pi(s+1)} + j_{\pi(s+1)} \quad (1 \leq s \leq r)$$

so by permuting with π we can make the powers of p weakly increasing. Then we can combine summands for the same power of p , this gives new coefficients of the form f_τ where each f_τ is a sum of some of the b_i , and $\sum f_\tau = \sum b_i$.

Explicitly, write $c_i = b_{\pi(i)}$, and write

$$(*) \quad \left(\sum a_i p^{j_i} = \right) \sum_i c_i p^{u_{\pi(i)} + j_{\pi(i)}} = \sum_{\tau} f_{\tau} p^{s_{\tau}}$$

where f_{τ} is the sum of all c_i such that $u_{\pi(i)} + j_{\pi(i)} = s_{\tau}$ and where the s_{τ} are strictly increasing. Then (s_{τ}) belongs to the set $\mathcal{N}_d(f_{\tau})$. Now note that $\sum f_{\tau} = \sum_i c_i < l$. By induction, the size of $\mathcal{N}_d(f_{\tau})$ is bounded in terms of l , independent of d . Hence for each permutation π the set of expansions with (\dagger) is bounded in terms of n . The number of permutations needed is at most $r!$ and $r \leq n$, hence the size of $\mathcal{N}_d(a_i)$ is bounded in terms of n . \square

Corollary 7.1.2. *Fix both i and p . Then there is an upper bound for the dimension of $\text{Ext}^i(k, \Delta(a))$ for any $a \in \mathbb{N}$.*

7.2. These results lead to several intriguing questions. Firstly, it would be interesting to determine the general behaviour of the Ext groups and not just cohomology. In the SL_2 case, we would not expect the behaviour to be much different. This is because Ext groups are essentially the same as cohomology. Recall from [7, corollary 5.2] for $a, b, i \in \mathbb{N}$ and $0 \leq i \leq p - 2$:

$$\text{Ext}^m(\Delta(pa + i), \Delta(pb + i)) \cong \begin{cases} \text{Ext}^m(\Delta(0), \Delta(p(b - a))) & \text{if } b - a \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\begin{aligned} & \text{Ext}^m(\Delta(pa + i), \Delta(pb + p - i - 2)) \\ & \cong \begin{cases} \text{Ext}^{m-1}(\Delta(0), \Delta(p(b - a - 1))) \oplus \text{Ext}^m(\Delta(a), \Delta(b - 1)) & \text{if } b - a \text{ is odd} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus for weights $pa + i, pb + i$ with even difference, their Ext group is a cohomology group. For weights $pa + i, pb + p - i - 2$ with odd difference, their Ext is almost a cohomology group but with an “error term” $\text{Ext}^m(\Delta(a), \Delta(b - 1))$. But this group is either zero, or can be written in terms of lower cohomology groups inductively. While we have not explored this further, this should not affect the growth rate significantly compared to the growth in $\text{Ext}^{m-1}(k, \Delta(p(b - a - 1)))$ so we would not expect significantly different results for more general Ext groups between Weyl modules.

This phenomenon of being essentially determinable from cohomology groups may not continue for more general algebraic groups. So it’s still possible for greater than exponential growth over all the Ext groups between Weyl modules for larger algebraic groups.

We may also consider bounding Ext groups between Weyl modules for more general algebraic groups. I.e. is there a bound $c \in \mathbb{N}$ for which $\dim \text{Ext}_G^i(\Delta(\lambda), \Delta(\mu)) \leq c$ for fixed i and any λ, μ a dominant weight? We have no feeling for this, but note that the bounds that work for SL_2 are known not to work for SL_3 and hence for larger algebraic groups. Although all the Hom spaces

are 1-dimensional for SL_3 , there are 2-dimensional Ext^1 groups between Weyl modules. Also, it is known that there are larger than 1-dimensional Hom spaces between Weyl modules for larger algebraic groups. Thus the question of boundedness is very much wide open.

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